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PARTICLE MOTION IN A PLASTIC MEDIUM WITH ALLOWANCE FOR PARTICLE ATTRITION

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Mathematical simulation of impact-induced deep penetration of an absolutely rigid spherical particle into an ideally plastic medium is performed; the law of particle motion and the distance covered by the particle are determined. The problem for a particle whose size varies owing to attrition is solved.

Key words: *plastic medium, penetration, wear of a spherical particle, mathematical model.*

Implantation of solid particles into plastic materials is studied in a large number of papers (see, e.g., [1–4]). In known solutions, however, the process of particle penetration into a plastic half-space was considered without allowance for the change in particle size due to its attrition. In the present paper, mathematical simulation of the process of deep penetration of a spherical particle into an ideally plastic medium is performed and the problem of particle wear is solved. The variational principle and the phenomenological theory of destruction described in [5] are used. Particle entry into the plastic medium is accompanied by the formation of a crater; during subsequent motion of the particle, the entry orifice collapses. During particle attrition, its mass remains in the plastic medium; the total volume of the particle and the medium remains unchanged. We consider the motion of the spherical particle at the stage when the particle has penetrated rather deeply into the plastic medium. In solving the boundary-value problem, it was assumed that a plastic-flow region is formed around the particle, and this region also takes a spherical form as the particle penetrates into the plastic medium (Fig. 1). The assumption of the spherical flow region around the particle was used because propagation of disturbances outside the boundaries of the plastic region cannot be significant, which is confirmed experimentally [6–9]. The flow region around the particle moves together with the particle. Note, the size of the plastic-flow region is determined by solving the corresponding variational problem.

In solving the boundary-value problem, the principle of virtual velocities and stresses is used as the basic one. Integration of differential equations of the boundary-value problem of mechanics of continuous media is replaced by an equivalent solution on virtual states of the variational equation [10].

In solving the boundary-value problem, it is assumed that the solid body moving with a velocity $v(t)$ and the plastic-flow region localized around the body are attached to a spherical coordinate system (r, φ, θ), which moves with a velocity v in the reference system (x, y, z). The particle radius decreases in accordance with the accepted wear model. The kinematically admissible pattern of the flow around a solid spherical particle by a plastically deformable medium is shown in Fig. 2.

The specific feature of motion of the particle and the plastic-flow region localized around the particle is that the moving (in the reference system) boundary S_0 of the plastic-flow region is the discontinuous surface of the tangential component of the velocity vector, i.e., the surface of a strong discontinuity (in the present dynamic problem, the wave of a strong discontinuity [11]). The normal component of the velocity vector of material particles of the plastically deformable medium in passing through the boundary S_0 is continuous. In the plastic-flow region, the medium flows around the solid spherical particle on all sides. On the front surface of the particle $r = R$ $(-\pi/2 \leqslant \theta \leqslant \pi/2)$, the shear stresses equal the yield point of the medium in shear: $\sigma_{\rho\theta} = \tau_s$. On the back surface of the particle $r = R$ $(\pi/2 \leqslant \theta \leqslant 3\pi/2)$, spalling of the medium is possible; therefore, tangential stresses on this surface can be zero.

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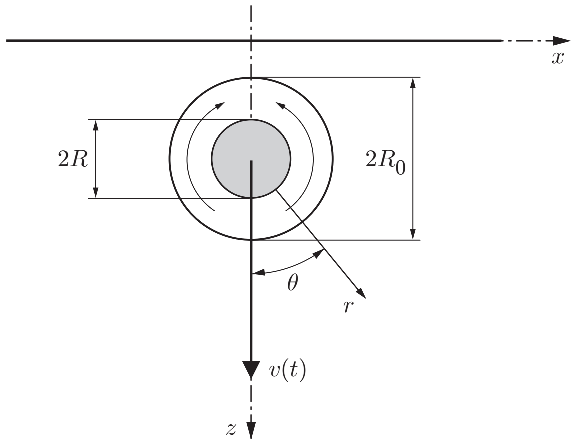

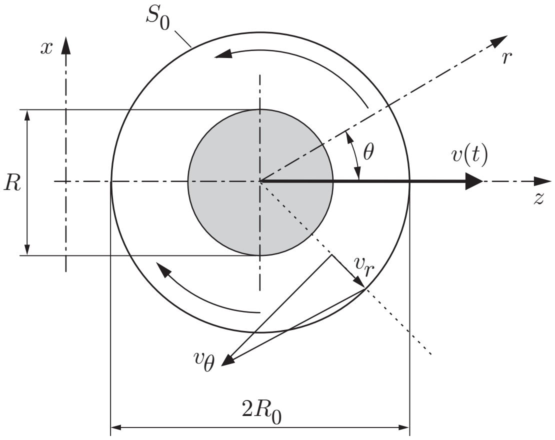

Fig. 1

Fig. 2

Fig. 1. Pattern of particle motion in a plastic medium.

Fig. 2. Pattern of the flow around a solid particle by a plastic medium.

Using the boundary conditions

$$v_r\Big|_{r=R_0} = 0, \qquad v_r\Big|_{r=R} = v(t)\cos\theta,$$

$$v_r\Big|_{\theta=\pm\pi/2} = 0, \qquad v_\theta\Big|_{\theta=0,\ \theta=\pi} = 0$$

for the plastic-flow region, we introduce the kinematically admissible velocity field

$$v_r = v(t)\frac{1}{\alpha^2-1}\Big(\alpha^2\,\frac{R^2}{r^2} - 1\Big)\cos\theta, \qquad v_\theta = v(t)\,\frac{1}{\alpha^2-1}\,\sin\theta, \qquad v_\varphi = 0,$$

where $\alpha = R_0/R$ is the varied parameter depending on the position of the plastic-region boundary.

The velocity field of the solid particle in the spherical coordinate system is

$$v_r = v(t)\cos\theta, \qquad v_\theta = -v(t)\sin\theta, \qquad v_\varphi = 0.$$

Taking into account that $v_\varphi = 0$, $v_r \neq v_r(\varphi)$, and $v_\theta \neq v_\theta(r,\varphi)$, we write the components of the acceleration vector in the spherical coordinate system as follows:

$$w_r = \frac{\partial v_r}{\partial t} + v_r\,\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\,\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r},$$

$$w_\theta = \frac{\partial v_\theta}{\partial t} + \frac{v_\theta}{r}\,\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}, \qquad w_\varphi = 0.$$

The strain rates in the spherical coordinate system have the form

$$\xi_{rr} = \frac{\partial v_r}{\partial r}, \qquad \xi_{\theta\theta} = \frac{1}{r}\,\frac{\partial v_{\theta\theta}}{\partial\theta} + \frac{v_r}{r}, \qquad \xi_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r} + \cot\,\theta\,\frac{v_\theta}{r},$$

$$\xi_{r\varphi} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\,\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi}{r} + \frac{\partial v_\varphi}{\partial r}\Big), \qquad \xi_{r\theta} = \frac{1}{2}\Big(\frac{1}{r}\,\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\Big),$$

$$\xi_{\theta\varphi} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\,\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{r}\,\cot\,\theta + \frac{1}{r}\,\frac{\partial v_\varphi}{\partial\theta}\Big).$$

For the velocity field introduced, after computations, we obtain

$$\xi_{rr} = -2v(t)f(r,\alpha)\cos\theta, \qquad \xi_{\theta\theta} = \xi_{\varphi\varphi} = v(t)f(r,\alpha)\cos\theta,$$

$$\xi_{r\varphi} = \xi_{\theta\varphi} = 0, \qquad \xi_{r\theta} = -(1/2)v(t)f(r,\alpha)\sin\theta,$$

where $f(r,\alpha) = \alpha^2 R^2/((\alpha^2-1)r^3)$. The volume of the medium does not change in the plastic-flow region:

$$\xi = \xi_{rr} + \xi_{\theta\theta} + \xi_{\varphi\varphi} = 0.$$

The intensity of the shear strain rate is determined by the expression

$$H = \sqrt{(2/3)[(\xi_{rr}-\xi_{\theta\theta})^2 + (\xi_{\theta\theta}-\xi_{\varphi\varphi})^2 + (\xi_{\varphi\varphi}-\xi_{rr})^2] + 4(\xi_{r\theta}^2 + \xi_{\theta\varphi}^2 + \xi_{\varphi r}^2)}.$$

For the chosen velocity field, we have

$$\xi_{\theta\theta} = \xi_{\varphi\varphi} = -\xi_{rr}/2, \qquad \xi_{r\varphi} = \xi_{\theta\varphi} = 0.$$

Then,

$$H = 2\sqrt{3\xi_{\theta\theta}^2 + \xi_{r\theta}^2} = v(t) f(r,\alpha)\sqrt{12\cos^2\theta + \sin^2\theta} \approx 2\sqrt{3}v(t)f(r,\alpha)|\cos\theta|.$$

To determine the radius of the plastic-flow region, we solved a variational problem with the use of the variational equation of the principle of virtual velocities and stresses. Taking into account the above-mentioned features of the kinematically admissible velocity field and neglecting the specific mass forces, we obtain

$$I = \int\limits_{\Omega_0} \tau_s H'\,d\Omega + \int\limits_{\Omega_0} \rho_0 w_i v_i'\,d\Omega + \int\limits_{\Omega_1} \rho_1 w_i v_i'\,d\Omega$$

$$+ \int\limits_{S_0} \tau_s[\Delta v_\theta']\Big|_{r=R_0} dS_0 + \int\limits_{S_1} \tau_s[\Delta v_\theta']\Big|_{r=R} dS = \sum_{i=1}^{5} I_i, \tag{1}$$

where

$$[\Delta v_\theta']\Big|_{r=R_0} = v(t)R^2\sin\theta/(R_0^2 - R^2) = v(t)\sin\theta/(\alpha^2-1),$$

$$[\Delta v_\theta']\Big|_{r=R} = v(t)R_0^2\sin\theta/(R_0^2 - R^2) = v(t)\alpha^2/(\alpha^2-1)\sin\theta,$$

$[\Delta v_\theta]$ is the difference in the tangential component of the velocity vector on the surfaces S_0 and S_1 bounding the plastic-flow region and the solid spherical particle; Ω_0 and Ω_1 are the volumes of the spherical layer (for the virtual mass moving together with the particle) and particle, respectively.

The components of acceleration in the functional are not varied; therefore, to eliminate indeterminacy in calculating the functional I_2, they are represented in the form of the following functions:

$$w_r = w_r\Big(a_*, \frac{R}{r}, \frac{dv}{dt}\cos\theta\Big), \qquad w_\theta = w_\theta\Big(a_*, \frac{dv}{dt}\sin\theta\Big), \qquad w_\varphi = 0.$$

Here $\alpha_* = R_*/R$, in contrast to $\alpha = R_0/R$, is not a varied parameter. It is assumed that α_* characterizes the position of the real boundary of the plastic-flow region.

As a result, the following expressions are obtained:

$$I_1 = \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_R^{R_0} \tau_s H r^2\,dr\,\sin\theta\,d\theta\,d\varphi \approx 4\sqrt{3}\,\pi\tau_s v(t)R^2\,\frac{\alpha^2}{\alpha^2-1}\ln\alpha,$$

$$I_2 = \rho_0\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_R^{R_0} w^i v_i r^2\,dr\,\sin\theta\,d\theta\,d\varphi$$

$$= \frac{4}{3}\,\pi R^3\rho_0\frac{\alpha^2}{\alpha^2-1}\,\frac{\alpha_*^2}{\alpha_*^2-1}\Big[\Big(1-\frac{1}{\alpha}\Big)^2 + \frac{1}{\alpha_*^2}\Big(1-\frac{1}{\alpha^2}\Big)\Big]v(t)\,\frac{dv}{dt} \qquad (\alpha_* > 1),$$

$$I_3 = \rho_1\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_R^{R_0} w^i v_i r^2\,dr\,\sin\theta\,d\theta\,d\varphi = \rho_1\,\frac{4}{3}\,\pi R^3 v(t)\,\frac{dv}{dt},$$

$$I_4 = \int_0^{2\pi}\int_0^{\pi} \tau_s[v_\theta]\Big|_{\rho=R_0} R_0^2 \sin\theta \, d\theta \, d\varphi = \pi^2 \tau_s v(t) R^2 \, \frac{\alpha^2}{\alpha^2-1},$$

$$I_5 = \int_0^{2\pi}\int_0^{\pi} \tau_s[v_\theta]\Big|_{\rho=R} R^2 \sin\theta \, d\theta \, d\varphi = \pi^2 \tau_s v(t) R^2 \, \frac{\alpha^2}{\alpha^2-1}.$$

After calculating the integrals, the functional I is represented in the form of a dimensionless function of the varied parameter α

$$\tilde{I} = \frac{\alpha^2}{\alpha^2-1}\Big\{4\ln\alpha + \frac{2}{\sqrt{3}}\,\pi - \frac{\alpha_*^2}{\alpha_*^2-1}\Big[\Big(1-\frac{1}{\alpha}\Big)^2 + \frac{1}{\alpha_*^2}\Big(1-\frac{1}{\alpha^2}\Big)\Big] q_0(t)\Big\} - q_1(t), \qquad (2)$$

where

$$q_0 = -\frac{4}{3\sqrt{3}}\,\frac{\rho_0}{\tau_s}\,R\,\frac{dv}{dt} \geqslant 0, \qquad q_1 = -\frac{4}{3\sqrt{3}}\,\frac{\rho_1}{\tau_s}\,R\,\frac{dv}{dt} > 0.$$

If the densities of the deformable medium ρ_0 and the particle ρ_1 are identical, then $q_1 = q_0 = q$.

The functional $\tilde{I}$ with varied α should take the minimum value. In this case, the condition $\partial\tilde{I}/\partial\alpha = 0$ is satisfied. From the source theorems [1], it follows that, in a state close to the real one, we have

$$\tilde{I}[\alpha_*] = \inf_{q(dv/dt)\geqslant 0} \tilde{I}(\alpha) \approx 0. \qquad (3)$$

It is also known that $\alpha = \alpha_*$ in the real state.

Thus, we have three equations for determining three unknowns entering into Eq. (2). It should be noted that q_0 and q_1 in (2) are functions of one parameter, namely, the acceleration $w(t) = dv/dt$.

The resultant system of equations is solved by the method of successive approximations with the use of an iterative procedure.

The system is solved most simply if the mechanical characteristics (yield point and density) of the medium coincide with those of the implanted particle. Then, functional (1) takes the form

$$\tilde{I}(\alpha) = F_1(\alpha)[F_2(\alpha) - F_3(\alpha_*, \alpha) q(t)] - q(t), \qquad (4)$$

where

$$F_1(\alpha) = \frac{\alpha^2}{\alpha^2-1}, \quad F_2(\alpha) = 4\ln\alpha + \frac{2}{\sqrt{3}}\,\pi, \quad F_3(\alpha_*, \alpha) = \frac{\alpha_*^2}{\alpha_*^2-1}\Big[\Big(1-\frac{1}{\alpha}\Big)^2 + \frac{1}{\alpha_*^2}\Big(1-\frac{1}{\alpha^2}\Big)\Big].$$

For $\alpha = \alpha_*$ (which corresponds to the real state), with allowance for Eq. (3), we can assume that $\tilde{I}(\alpha_*) = 0$. Then, it follows from Eq. (4) that

$$q(t) \approx q(\alpha_*) = F_2(\alpha_*)/(F_3(\alpha_*, \alpha_*) + 1/F_1(\alpha_*)).$$

Substituting this result into (4), we obtain the functional of one varied parameter

$$\tilde{I}(\alpha) = F_1(\alpha)[F_2(\alpha) - F_3(\alpha_*, \alpha) q(\alpha_*)] - q(\alpha_*).$$

The variational problem considered has a solution that satisfies all required conditions, which is confirmed by the calculation results shown in Fig. 3. Among the set of curves $\tilde{I}(\alpha, \alpha_*) = \tilde{I}(\alpha)|_{\alpha_*=\text{const}}$, only one has a minimum corresponding to the point $\alpha = \alpha_*$, and $\tilde{I}(\alpha_*, \alpha_*) = 0$.

If the densities of the implanted particle and the medium are identical, it follows from the results obtained that $\alpha_* = R_0/R = 3$ and $q_* = q(\alpha_*) = 5.344$. Thus, the specific deformation force decelerating particle motion in the dynamic problem is 1.6 times smaller than the force necessary for particle motion with a constant velocity.

Knowing the values of $\alpha_* = R_*/R$ and $q_* = q(\alpha_*)$, from the equation

$$q_* = -\frac{4}{3\sqrt{3}}\,\frac{\rho_1}{\tau_s}\,R\,\frac{dv}{dt}$$

we find the relation for calculating the acceleration of the solid particle

$$w(t) = \frac{d^2 z}{dt^2} = -\frac{3\sqrt{3}}{4}\,q_*\,\frac{\tau_s}{\rho_1 R}.$$

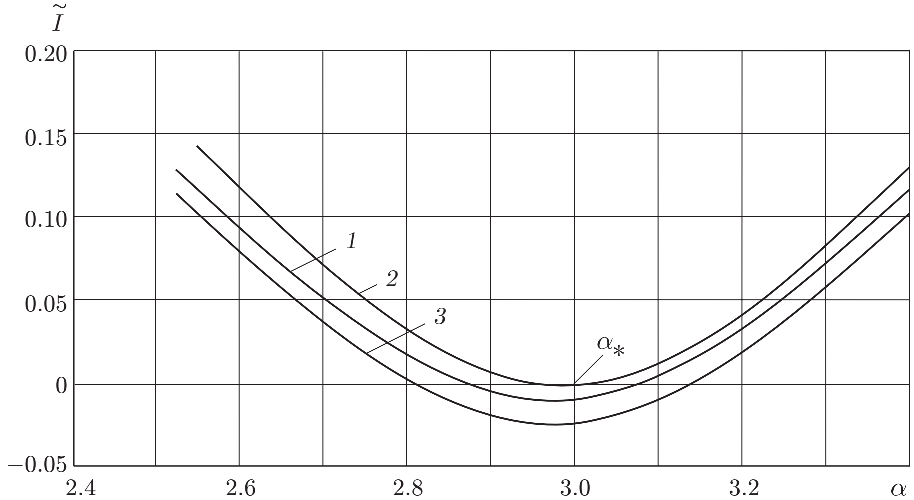

Fig. 3. Results of the solution for $\alpha_* = 2.8$ (1), 3 (2), and 3.2 (3).

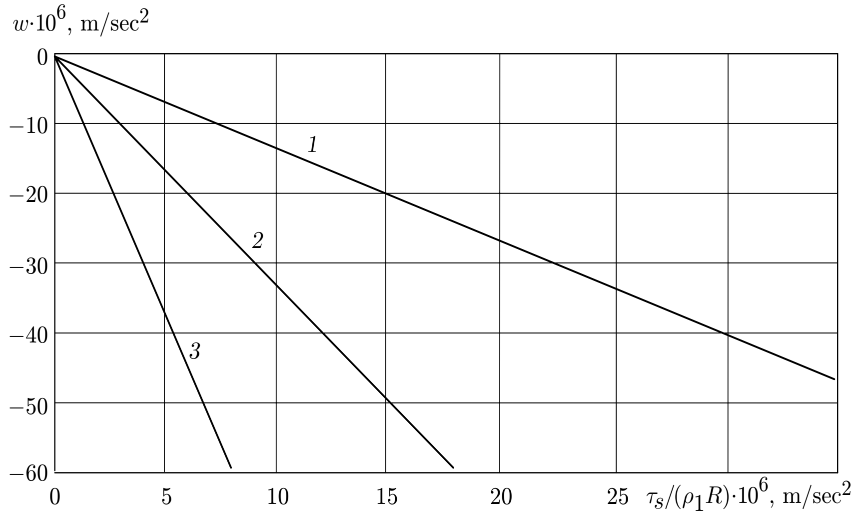

Fig. 4. Results of the solution for $q_* = 1$ (1), 2.5 (2), and 5.344 (3).

The calculation results are plotted in Fig. 4. The calculations were performed for the case of identical densities of the solid particle and the plastically deformed medium ($\rho_1 = \rho_0$).

We calculate the distance covered by the particle before it stops completely by solving the equation

$$u_z = z_m - z_0 = v_1^2/(2Q),$$

where $Q = (3\sqrt{3}/4)q_*\tau_s/(\rho_1 R)$, with the initial conditions $\dot{z}(0) = v_1$ and $z(0) = z_0$.

As an example, we determine the distance u_z covered by the solid particle of radius $R = 10^{-3}$ m with an initial velocity $v_1 = 2000$ m/sec in a medium with a yield point in shear $\tau_s = 300$ MPa for $\rho_1 = \rho_0 = 7800$ kg/m^3. In the course of the solution, we obtain the value $u_z = 3.74 \cdot 10^{-3}$ m, i.e., $u_z = 3.74R$.

We determine the influence of variation of the particle radius on the distance covered by the particle. If the particle is not absolutely solid, then it experiences the action of the friction force and is subjected to wear when moving in a continuous medium. The particle mass and size decrease. Obviously, the change in the mass and size of the moving particle can exert a significant effect on the law of its motion and on the distance covered by the particle. Therefore, we have to consider some of the existing models of friction and wear.

TABLE 1

i	v_i, m	Δu_{zi}, mm	R_i, mm	u_{zi}, mm
1	1900	0.703	1.000	0.703
2	1700	0.624	0.992	1.327
3	1500	0.551	0.985	1.878
4	1300	0.473	0.980	2.351
5	1100	0.399	0.975	2.750
6	900	0.325	0.971	3.075
7	700	0.251	0.968	3.326
8	500	0.179	0.967	3.505
9	300	0.107	0.966	3.612
10	100	0.036	0.966	3.648

The model of contact interaction of rubbing surfaces in the case of slipping friction was proposed in [12], where it was assumed that contact stresses are a consequence of interaction of wedge-shaped roughness elements of rubbing surfaces. As the normal stresses increase and the surfaces are shifted with respect to one another, new roughness elements come into contact, which actually increases the contact area, which cannot exceed a certain fraction of the contact-surface area. At a certain stage of the process, the increase in the actual contact area become slower, which slows down the growth of contact shear stresses with a further continuous increase in normal stresses p_n. In the model considered, the contact stresses have a certain limit. Such a model agrees with the results of theoretical and experimental studies [13–17] and others. In the present work, the action of dry friction with limiting high normal pressures is assumed. In this case, it is possible to use the known Prandtl–Siebel friction law

$$\tau_c = f\tau_s,$$

where τ_s is the yield point of the deformed medium; the coefficient of proportionality f takes the values 0 or 1 ($f = 1$ at the front surface of the particle, where the normal compression stresses act).

In the case of surface friction, the basic form of wear is attrition due to the fatigue state of the contact surface. The intensity of such wear increases with increasing normal and shear contact stresses and decreases with increasing velocity of displacement of rubbing surfaces. These factors are taken into account in the wear model of [18] in which the wear intensity $I_h = dh/dS$ (h is the linear wear and S is the relative displacement of the solid particle and plastically deformed medium) is proportional to normal pressure and shear stress, increases with increasing temperature, and is inversely proportional to the velocity of relative slipping and hardness of the material being processed. The mathematical model of boundary friction and wear for heavily loaded slipping pairs is developed in [19, 20].

It should be noted that the existing models describe the processes of wear during boundary friction, i.e., it is assumed that there is some lubrication that partly separates the rubbing surfaces. In the process considered, we have dry friction accompanied by adhesion interaction of contact surfaces and their attrition. In this case, for calculating the change in the particle radius R during particle motion along the z axis (see Fig. 1), we use the differential equation

$$\frac{dR}{dz} = k_0\Big(\frac{\tau_s}{\tau_{s1}} + k_1\,\frac{v}{v_*}\Big),$$

where τ_{s1} is the yield point of the particle, v/v_* is the relative velocity of the particle ($v_* = 1000$ m/sec), and k_0 and k_1 are empirical coefficients.

To determine the change in the radius of the moving particle, we have the following equation in finite differences:

$$R_{i+1} = R_i - k_0(\tau_s/\tau_{s1} + k_1 v_i/v_*)\Delta u_{zi}, \qquad i = 1, 2, 3, \ldots, n.$$

Here Δu_{zi} is the distance covered by the particle at the ith step and n is the total number of time steps simulating particle motion.

To find Δu_{zi}, we use the equation

$$\Delta u_{zi} = \frac{2}{3\sqrt{3}}\Big(n - i + \frac{1}{2}\Big)\Big(\frac{v_1}{n}\Big)^2\frac{\rho_1 R_{i-1}}{q_*\tau_s}.$$

The distance covered by the particle until its complete stop is determined as

$$u_z = \sum_{i=1}^{n} \Delta u_{zi}.$$

As an example, we determine the distance covered by the solid particle ($R = 10^{-3}$ m, $v_1 = 2000$ m/sec) in a medium with a yield point in shear $\tau_s = 300$ MPa for $\rho_1 = \rho_0 = 7800$ kg/m^3 and $\tau_{s1} = \tau_s$. The empirical coefficients have the values $k_0 = 10^{-2}$ and $k_1 = 10^{-1}$. The calculation results are listed in Table 1.

As a result of calculations, we obtained the value $u_z = 3.65 \cdot 10^{-3}$ m. Thus, the particle whose diameter varies as a result of attrition covers a smaller distance before its complete stop than the absolutely solid particle. The ratio of the covered path to the particle radius increases by 1%.

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